المقاسات الاولية–الضاهرية الضعيفة الجوهرية

بحث مقدم من قبل

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Essential Weakly Quasi-Prime Modules

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Abstract

Let R be a commutative ring with identity, and M be a left unitary R-module. In this paper, we introduce and study type of modules namely essential weakly quasi-prime module as ageneralization of prime module. An R-module M is called essential weakly quasi-prime module if $ann_R U$ is weakle prime ideal for each anon-zero essential submodule U of M, "where a submodule U of M is called essential if$U \cap T \neq (0)$ for anon-zero submodule T of M". In this paper, we have presented some characteristics of this concept, as well as the relationship of this concept with other classes of modules.

Keywords: Essential Weakly Quasi-Prime Modules, "PrimeModules", "UniformModules", "Multiplication Modules", "Quasi-Prime Modules".

المستخلص

ليكن R حلقة أبدالية بمحايد, وليكن M مقاساَ أحادي أيسراً على الحلقةR . في هذا البحث قدمنا ودرسنا نوع من المقاسات وهي المقاسات الاولية الضاهرية الضعيفة الجوهرية التي هي تعميم للمقاسات الاولية. يدعى المقاس M مقاساَ اولياَ–ضاهرياَ ضعيفاَ جوهرياَ أذا كان $ann_R U$ مثالي أولي ضعيف , لكل مقاس جزئي جوهري غير صفري Uمن M. حيث يدعى المقاس الجزئي U من M جوهرياَ أذا كان $U \cap T \neq 0$, لكل مقاس جزئي غير صفري T من M.أن الهدف الاساسي من هذا البحث هو أعطى بعض الخصائص والصفات لهذا المفهوم وكذلك علاقة هذا المفهوم مع المقاسات الاخرى.

1- Introduction

Let R be a associative ring with identity, and all module are unitary left R-modules. "An R-module M is called a prime if $ann_R M = ann_R N$ for every anon-zero submodule N of M". Following [1]. Abdul-Razaak in [1] introduced and studied the concept of quasi-prime module as a generalization of prime module, "where an R-module M is called quasi-prime if and only if $ann_R N$ is a prime ideal of R, for every anon-zero submodule N of M". "An R-module M is called weakly prime if the annihilator of any non-zero submodule of M is aprime ideal of R" [3]. Tha'ar Y. G. in [4] the concept essential prime module as a generalization of prime module, "where an R-module M is called essential prime if $ann_R M = ann_R N$ for every essential asubmodule N of M". Mijbass A. S. in [7], "where an R-module M is called quasi-Dedekind if $Hom(M/N, M) = 0$ for all non-zero submodule N of M".

This leads us to introduce the concept of essential weakly quasi-prime module, where an R-module M is called essential weakly quasi-prime module if and only if $ann_R N$ is weakle prime ideal for each anon-zero essential submodule N of M. "An R-module M over principle ideal domain is divisible if $rM = M$ for every non-zero element $r \in R$" [1]. Moreover we give many relationships between essential weakly quasi-prime modules, prime modules, essential prime modules, quasi-Dedekind modules.

2-Essential Weakly Quasi-Prime Modules:

throughout this paper we introduce the definition of essential weakly quasi-prime module, give some basic, example and characterization of this concept.

Definition 2.1

An R-module M is called essential weakly quasi-prime if and only if $ann_R N$ is weakle prime ideal for each a non-zero essential submodule N of M. Aring R is called essential weakly quasi-prime if and only if R essential weakly quasi-prime R-module.

Example and Remark 2.2

1 – It is clear that every quasi-prime module is essential weakly qasi-prime module.
2 – Every "prime module" is essential weakly quasi-prime module by [2, Remark (1.2.2)] and by (1). But the convers is not true in general for example : The Z-module $Z \oplus Z_5$ is essential weakly quasi-prime Z-module, for each essential submodule N of $Z \oplus Z_5$ the $ann_z N$ is weakle prime ideal of Z. But $Z \oplus Z_5$ is not prime because $ann_z Z \oplus Z_5 = (0)$ and $ann_z Z \oplus Z_5 = 5Z \neq (0)$.
3 – since Q as a Z-module is prime, then Q is essential weakly quasi-prime Z-module.
4 – Z as a Z-module is essential weakly quasi-prime Z-module.
5 - $Z \oplus Z$ as a Z-module is essential weakly quasi-prime Z-module.
6 - Z_n as a Z-module is essentially weakly quasi-prime Z-module if and only if n is prime number.
7 – The homomorphic image of essential weakly quasi-prime R-module need not to be essential weakly quasi-prime R-module as the following example : The Z-module Z is essential weakly quasi-prime Z-module. But $\frac{Z}{8Z} \cong Z_8$ is not essential weakly quasi-prime Z-module.
8 -The direct sum of two essential weakly quasi-prime R-module need not to by essential weakly quasi-prime R-module as the following example shows : The Z-module $Z_3 \oplus Z_5$ is not essential weakly quasi-prime Z-module, because $ann_Z(Z_3 \oplus Z_5) = 15Z$ is not weakly prime ideal of Z, while Z_3 and Z_5 are essentially weakly quasi-prime Z-module.
9 – Every essential submodule of essential weakly quasi-prime R-module is essential weakly quasi-prime R-module.
Proof: Let M be essential weakly quasi-prime R-module and N is essential submodule of M. Let K is essential submodule of N, that is K is essential submodule of M by [3, prop. 5.16 , p.74].

But M is essentially weakly quasi-prime, then $ann_R K$ is weakly prime ideal of R. Hence N is essential weakly quasi-prime of M.

10 – Every "weakly prime R-module" is essential weakly quasi-prime R-module.

11 - Every essential prime R-module is essential weakly quasi-prime R-module.

12 – Every simple R-module is essential weakly quasi-prime R-module.

Proposition 2.3

Let M be an R-module, then the following statement are equivalent :

1 - M is essential weakly quasi-prime R-module.

2 - $ann_R N = ann_R rN$ for each essential submodule N of M, such that $rN \neq 0$, $0 \neq r \in R$ and $ra \neq 0$ for each $0 \neq a \in R$.

3 - $ann_R(m) = ann_R(rm)$ for each essential submodule generated by $m \in M$ and $0 \neq r \in R, rm \neq 0$, and $ra \neq 0$ for each $0 \neq a \in R$.

4 - $ann_R(m)$ is weakly prime ideal for each a non-zero essential submodule generated by $m \in M$.

Proof :

(1) ⇨ (2) Let M be an essential weakly quasi-prime R-module, and N be a non-zero essential submodule of M, s.t $rN \neq (0)$, $0 \neq r \in R$ and $ar \neq 0$ for each $0 \neq a \in R$. Let $a \in ann_R(rN)$ then $arN \neq 0$ which implies that $ar \in ann_R N$. But M is essential weakly quasi-prime R-module then $ann_R N$ is weakly prime ideal of R. Hence either $a \in ann_R N$ or $r \in ann_R N$, if $r \in ann_R N$ then $rN = 0$ this is a contradiction. Therfore $a \in ann_R N$ thus $ann_R rN \subseteq ann_R N$.

On the other hand we have $rN \neq 0$, $0 \neq r \in R$ and $ra \neq 0$ for each $0 \neq a \in R$, since $rN \subseteq N$ that is $ann_R N \subseteq ann_R rN$.

(2) ⇨ (3) It is clearly , for each a non-zero essential submodule $N = (m)$ generated by $m \in M$, $0 \neq r \in R, rm \neq 0$, and $ra \neq 0$ for each $0 \neq a \in R$.

(3) ⇨ (4) Let $0 \neq ab \in ann_R(m)$, and suppose that $b \notin ann_R(m)$ for every non-zero $a, b \in R$ then $abm = 0$ and $bm \neq 0$, which implies that $a \in ann_R(bm)$. But $ann_R(m) = ann_R(rm)$ by hypothesis, then $a \in ann_R(m)$. That is $ann_R(m)$ is weakly prime ideal.

(4) ⇨ (1) suppose that $ann_R(m)$ is weakly prime ideal of R for each a non-zero essential submodule of M generated by $m \in M$, let N be a non-zero essentialsubmodule of M, then $ann_R(n) = [\,0 :_R (n)]$ is weakly prime ideal of R, for each essential submodule (n) generated by $n \in N$, then by [2, Lemma (1.2.5)] we get $[\,0 :_R N\,]$ is a weakly prime ideal of R. that is $ann_R N$ is weakly prime ideal of R, for each a non-zero essential submodule N of M.

Corollare 2.4

Let M be an essential weakly quasi-prime R-module then $ann_R M$ is weakly prime ideal of R.

The following proposition showe the direct sum of to essentially weakly quasi-prime R-module is essentially weakly quasi-prime R-module under a certion condition.

Proposition 2.5

Let M_1 and M_2 be two essentially weakly quasi-prime R-module such that for each essential submodule N and K of M_1 and M_2 respectively, and either $ann_R N \subseteq ann_R K$ or $ann_R K \subseteq ann_R N$. Then $M = M_1 \oplus M_2$ is essential weakly quasi-prime R-module.

Proof :

Since N is a non-zero essential submodule of M_1 and K is essential submodule of M_2 then by [5, Coro. (5.1.7)], we have $N \oplus K$ is essential submodule of M, we must prove that $ann_R(N \oplus K)$ is a weakly prime ideal of R. But we have $ann_R(N \oplus K) = ann_R N \cap ann_R K$, since either

$ann_R N \subseteq ann_R K$ or $ann_R K \subseteq ann_R N$, so either $ann_R(N \oplus K) = ann_R N$ or $ann_R(N \oplus K) = ann_R K$. But M_1 and M_2 is essential weakly quasi-prime R-module, the $ann_R N$ and $ann_R K$ are a weakly prime ideal of R. Therefore $ann_R(N \oplus K)$ is a weakly prime ideal of R.Thus $M_1 \oplus M_2$ is essentially weakly quasi-prime R-module.

3 - <u>Essential weakly quasi-prime module and related concepts</u> :

In this section we study the relationships between essential weakly quasi-prime module and other classes of modules.

"Recall that an R-module M is called uniform, if every submodule of M is essential in M" [6].

<u>Proposition 3.1</u>

Let M be a uniform Rmodule. Then M is essential weakly quasi-prime R-module if and only if M is prime.

<u>Proof</u> :

Let M be essential weakly quasi-prime R-module, we must prove that $ann_R M = ann_R(m)$ for every $0 \neq m \in M$. It is clear that $ann_R M \subseteq ann_R(m)$. Let $x \in ann_R(m)$ then $x(m) = (0)$ and let $0 \neq m' \in M$, since M is uniform then $(m) \cap (m') \neq 0$. That is there exists $r_1, r_2 \in R$ such that $r_1 m = r_2 m' \neq 0$. But $xm = 0$, implies that $xr_1 m = 0$. It follows that $xr_2 m' = xr_1 m = 0$, hence $x \in ann_R(r_2 m')$. But M is essentially weakly quasi-prime R-module then by prop. (2.3), $x \in ann_R(m')$ that is $xm' = 0$ for each $0 \neq m' \in M$. Thus $x \in ann_R M$ and hence $ann_R M = ann_R(m)$. Therefore M is prime.

The convers is direct.

"Recall that an R-module M is called quasi-Dedekind R-module if $Hom_R(M/N, M) = (0)$ for all a non-zero submodule N of M" [7].

<u>Proposition 3.2</u>

Every quasi-Dedekind R-module is essential weakly quasi-prime R-module.

<u>Proof</u> :

Let M is quasi-Dedekind R-module then by [7], M is prime then M is essential weakly quasi-prime R-module.

<u>Corollare 3.3</u>

Let M is a uniform R-module. Then the following statements are equivalents :

1 - M is an essential weakly quasi-prime R-module.
2 - "M is prime R-module".
3 - "M is quasi-Dedekind R-module".

<u>Proof</u> :

(1) ⇨ (2) by Prop.(3.1).
(2) ⇨ (3) by [7, The. (3.11)].
(3) ⇨ (1) by Prop. (3.2).

"Recall that an R-module M is called essentialquasi-Dedekind R-module if $Hom_R(M/N, M) = (0)$ for each a non-zero essential submodule N of M" [4]

Proposition 3.4

Every essential quasi-Dedekind R-module is essential weakly quasi-prime R-module.

Proof :

Let N be essential submodule of M, since M is essential quasi-Dedekind R-module then N is quasi-invertible R-submodule then by [7, prop.(1.4)] $ann_R M = ann_R N$ that is M is essentially prime. Therefore M is essential weakly quasi-prime R-module.

In the following proposition, we can see that in the class of uniform modules, the next three concept are equivalent.

Proposition 3.5

Let M is a uniform R-module. Then the following statements are equivalents:

1 - M is an essential weakly quasi-prime R-module.

2 - "M is essential quasi-Dedekind R-module".

3 – "M is essential prime R-module".

Proof :

$(1) \Rightarrow (2)$ by Prop.(3.1) and [8, prop.(2.1.1)].

$(2) \Rightarrow (3)$ by [8, prop. (2.1.8)].

$(3) \Rightarrow (1)$ it is clear that.

"Recall that an R-module M is called bounded if, there exists $x \in M$ such that $ann_R M = ann_R(x)$ "[4].

Proposition 3.6

Let M be a bounded R-module with $ann_R M$ is weakly prime ideal of R. Then M is essential weakly quasi-prime R-module.

Proof :

Since M is bounded R-module, there exists $x \in M$ such that $ann_R M = ann_R(x)$. Let N be a essential submodule of M, hence there exists $0 \neq t \in R$ such that $0 \neq tx \in N$. To prove that $ann_R M = ann_R N$. It is clear that $ann_R(x) = ann_R M \subseteq ann_R N \subseteq ann_R(tx)$, let $r \in ann_R(tx)$, then $rtx = 0$ that is $rt \in ann_R(x)$. But $ann_R(x)$ is weakly prime ideal, so either $r \in ann_R(x)$ or $t \in ann_R(x)$. If $t \in ann_R(x)$, then $tx = 0$ which is a contradiction. Thus $r \in ann_R(x)$ that is $ann_R(tx) \subseteq ann_R(x)$. Thus $ann_R(x) = ann_R M \subseteq ann_R N \subseteq ann_R(x)$. Hence $ann_R M = ann_R N$. That is $ann_R N$ is weakly prime ideal of R. Therefore M is essential weakly quasi-prime R-module.

"Recall that an R-module M is called weakly quasi-Dedekind R-module if $Hom_R(M/N, M) = (0)$ for all quasi-essential submodule N of M" [8].

Proposition 3.7

Every weakly quasi-Dedekind R-module is essential weakly quasi-prime R-module.

Proof :

Let M be a weakly quasi-Dedekind R-module, then by [8, prop. (3.1.2)] M is weakly prime and by (2.1) M is essential weakly quasi-prime R-module.

Corollare 3.8

If M is a uniform bounded R-module with $ann_R M$ is a prime ideal of R. Then M is essential weakly quasi-prime R-module.

Proof :

By [8, prop. (3.1.11)] and prop. (3.7).

In the following proposition, we can see that in the class of uniform modules, the next four concept are equivalent.

Proposition 3.9

Let M be a uniform R-module. Then the following statements are equivalents:

1 - "M is quasi-Dedekind R-module".

2 – "M is essential quasi-Dedekind R-module".

3 - "M is weakly quasi-Dedekind R-module".

4 - M is an essential weakly quasi-prime R-module.

Proof :

(1) ⇨ (2) It isclear that for all essential submodule G of M.

(2) ⇨ (3) Since every essential submodule is quasi--essential submodule, then the prove direct.

(3) ⇨ (4) By prop. (3.7).

(4) ⇨ (1) By coro. (3.3).

Proposition 3.10

Let R be an integral domain, M be an R-module, if every submodule of M is divisible, then M is essential weakly quasi-prime R-module.

Proof :

By [1, prop. (1.5.3)], we get M is a prime R-module. Hence M is essential weakly quasi-prime R-module.

The converse of thus proposition is not true for example :

Z as a Z-module is essentially weakly quasi-prime Z-modul, it is clearly every non-zero submodule of Z is not divisible.

"Recall that a proper submodule N of M is called semi-prime if every $r \in R$, $r^n x \in N$, then $rx \in N$" [9].

Proposition 3.11

If M is divisible R-module an (0) is semi-prime submodule of M. Then M is essential weakly quasi-prime R-module.

Proof :

By [1, prop. (1.5.9)] M is prime, then M is essential weakly quasi-prime R-module.

"Recall that an R-module M is multiplication if every submodule N of M, there exists an ideal I of R such that $N = IM$" [10]

In the next propsition the two concepts prime and essential weakly quasi-prime module we can see that in the class of multiplication modules are equivalent.

Proposition 3.12

Let M be amultiplication R-module, such that for each $0 \neq r \in R$ and I is a non-zero ideal of R, $rI \neq 0$. Then M is essential weakly quasi-prime R-module if and only if M is prime.

Proof :

Let M be essential weakly quasi-prime R-module, and let $0 \neq r \in R$, $m \in M$ such that $r(m) = 0$, since M is multiplication , then $(m) = IM$ for some ideal I of R. Hence $rIM = 0$, that is $0 \neq rI \subseteq ann_R M$. But M is essential weakly quasi-prime R-module then $ann_R M$ is weakly prime ideal of R. Hence either $r \in ann_R M$ or $I \subseteq ann_R M$, thus either $r \in ann_R M$ or $IM = 0$. Hence either $r \in ann_R M$ or $(m) = 0$. Thus (0) is a prime submodule then by [11] M is prime R-module.

The conversely straight forward.

Recall that "an R-module M is called chaind R-module if the submodule of M are ordered by inclusion" [5].

Proposition 3.13

Let M be achain R-module. Then M is essential weakly quasi-prime R-module if and only if M is essentially prime R-module.

Proof :

Let M be essential weakly quasi-prime R-module and let $(m) = N$ be a non-zero essential submodule of M generated by $0 \neq m \in M$. To prove that $ann_R M = ann_R N$. It is clear that $ann_R M \subseteq ann_R N$. Let $a \in ann_R(m)$, then $am = 0$. Suppose that a $\notin ann_R(m)$, then there exists $0 \neq m_1 \in M$ such that $am_1 \neq 0$. But M is chain R-module then $(m) \subseteq (m_1)$ or $(m_1) \subseteq (m)$. If $(m_1) \subseteq (m)$, then $ann_R(m) \subseteq ann_R(m_1)$ implies that $am_1 = 0$ which is a contradiction. Thus $(m) \subseteq (m_1)$, that is $m = rm_1$ for some $0 \neq r \in R$ and $ar \neq 0$. Then $0 = am = arm_1$, implies that $a \in ann_R(rm_1)$. But M is essentially weakly quasi-prime R-module, then by (2.3) $a \in ann_R(m_1)$, that is $am_1 = 0$ which is a contradiction. Hence $a \in ann_R M$, thus $ann_R M = ann_R(m)$. That is M is essential prime R-module.

The other part by (2.2).

Recall that an R-module M is called cyclic R-module if there exist $x \in M$ such that $M = Rx$ [5].

Proposition 3.14

Let M be acyclic R-module. Then M is essential weakly quasi-prime R-module if and only if M is a quasi-Dedekind R-module.

Proof :

Let M be essential weakly quasi-prime R-module. Let $f \in End_R(M)$ such that $f \neq 0$. To show that f is monomrphism, let $M = (m)$ for some $0 \neq m \in M$. Now let $x \in kerf$, then $f(x) = 0$, but $x \in M = (m)$ then $x = r_1 m$ for some a non-zero element $r_1 \in R$. Hence $0 = f(x) = f(r_1 m) = r_1 f(m)$, but $f \neq 0$ then $f(m) \neq 0$ and $f(m) \in M$, so $f(m) = r_2 m$ for some non-zero element $r_2 \in R$. Hence $r_1 r_2 m = 0$ such that $r_1 r_2 \neq 0$ and $r_2 m \neq 0$, then $r_1 \in ann_R(r_2 m)$. But M is essentially weakly quasi-prime R-module, then by Prop. (2.3) $ann_R(m) = ann_R(r_2 m)$, for each essential submodule (m) of M, and $(m) = M$ is an essential in M. Hence $r_1 \in ann_R(m)$ which mean that $r_1 m = 0$. That is $x = 0$, which implies that $\ker f = \{0\}$, and hence f is monomorphism. Therefore M is quasi-Dedekind R-module by [7, Theorem (1.5)].

Conversely : by Prop. (3.7).

However, the following corollary shows that the three concept , essentail weakly quasi-prime modules, quasi-Dedekind modules and prime modules are equivalents in the class of cyclic module.

Corollare 3.15

Let M be acyclic R-module. Then the following statements are equaivalents :

1 - M is an essential weakly quasi-prime R-module.

2 - M is quasi-Dedekind R-module.

3 - M is prime R-module .

Proof :

It isclear.

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